Dynamic Competition with Network Externalities: Why History Matters*
(PRELIMINARY AND INCOMPLETE)

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Abstract

This paper considers dynamic platform competition in a market with network externalities. A platform that dominated the market in the previous period becomes "focal" in the current period, in that agents play the equilibrium in which they join the focal platform whenever such equilibrium exists. We ask whether a low-quality but focal platform can maintain its focal position along time, when it faces competition by a non-focal but higher quality platform. We find that the low-quality platform can maintain its focal position infinitely if the firms are impatient. When firms are patient, there are multiple equilibria in which either the low or the high quality platform dominates. If qualities are stochastic, an increase in the discount factor can make it easier for a low-quality platform to win. As a result, with stochastic qualities social welfare can decrease the more platforms care about future profits.

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1 Introduction

Platform competition typically involves repeated interaction. In each of the markets for smartphones, tablets, video-game consoles, etc., a small set of firms compete with each other repeatedly over time. Platforms should therefore take into account how their strategies today affect their future profits. How the competition in dynamic setting plays out may have an important effect on platforms' profits. Microsoft's Windows wins the market for computer operating system over Apple's OS many generations in a row. It has been often suggested that the Apple's OS is of a better quality, but Windows wins because Microsoft gained the dominant position in the past, and the network effects allow this advantage to carry over time, despite inferior product. In other markets, like video-game consoles and smartphones, market leaders seem to be changing every few generations. So the platforms in those markets cannot count on the same future advantage from winning the market as in the market for computer operating systems.

In repeated interaction between firms, dynamic considerations play a new role when the firms are "platforms," because in such markets, the firms operate in environments with network effects. A platform offers users a product which has some stand-alone value, but the value of the product increases if other users also join the same platform. The benefit may come directly from the presence of other users, or through endogenous provision of complementary goods (e.g., apps are more likely to be developed for a popular platform). The difficulty, however, is that users need to form beliefs before purchase about which platform will offer most network benefits. In many markets we observe that the platform that became the dominant in the recent past has the advantage of users' expectations that it will attract other users and/or complementor providers (e.g., app developers). That is, it becomes the *focal* platform. But despite this beliefs advantage, the platform that won in the past not necessarily will win in the future.

This paper considers repeated platform competition in a market with network externalities. We explore the implications of history dependency, when past success raises the chance to become focal, regarding two related research questions. First, in some cases, a platform that benefits from the focal position, can dominate the market even if it offers a product of lower stand-alone quality than a new platform. In such a case, the focal platform uses its focal advantage to overcome its quality disadvantage. In a dynamic environment, however, when platforms have an infinite horizon, we may expect that the platform with the highest quality will have the strongest incentive to compete aggressively in order to gain and than maintain a focal position. We therefore ask whether a low-quality platform can maintain its focal position along time, when facing a higher-quality platform.

The first research question has an important implication for social welfare. If a dynamic environment makes it more profitable for a high-quality platform to gain a focal position from a lower-quality platform and maintain it along time, then social welfare should increase the more platforms care about future profits. Our second, related, research question is therefore whether social welfare increases or decreases the more patient are the platforms.

To investigate these two research questions, we consider two platforms competing for infinite number of periods. In each period, one of the platforms wins the market. In order to focus on the dynamic aspects of the model, we assume homogeneous consumers. Hence the winning platform captures the whole market. The dynamic set-up allows consumers to base their behavior in the current period on the observation of the past outcomes. Specifically, the platform that won the market in the previous period becomes focal in the current period. In such a case, winning the market in one period gives the platform an advantage in the future periods. Hence, a non-focal platform may be willing to sacrifice current profit to gain future market position.

In our base model, we assume that each platform has stand-alone quality which is constant for all periods. We show that when the platforms are short-sighted (i.e., their discount factor is below a threshold value), the focal platform maintains its focal position even though it offers a lower quality than the non-focal platform, as long as the quality gap is sufficiently small. However, when platforms have a high discount factor and the quality gap is sufficiently small, there are multiple equilibria. Now, the focal and low-quality platform can maintain its focal position infinitely, but there is also an equilibrium in which the non-focal and high-quality platform wins the focal position and then maintain it infinitely. When platforms have intermediate discount factor and the quality gap is large, there is a unique equilibrium in which the high-quality platform wins the market even when it is not focal, and then maintain the focal position infinitely.

The intuition for these results is that when platforms are very short-sighted, a non-focal platform will not have an incentive to compete aggressively and incur loses to gain a focal position in the future, making it possible for the focal platform to maintain its focal position even with a lower quality. In contrast, when both platforms have a high discount factor, they will have a strong incentive to win and maintain a focal position. In such a case, the low-quality platform can win the market when the high-quality platform expects that even if it were to gain the focal position, the low-quality platform will compete aggressively in every period in order to gain the focal position back, making it not worthwhile for the high-quality platform to gain the focal position in the first place. For the same reason, there is also an equilibrium in which the non-focal and high-quality platform wins and maintain the focal position, because the low-quality platform expects that the non-focal and high-quality platform will compete very aggressively in every period in order to gain the focal position, making it not worthwhile for the low-quality platform to maintain its focal position. For intermediate discount factor and when the quality gap is high, the former equilibrium fails and there exists a unique equilibrium in which the non-focal and high-quality platform gains and maintains the focal position.

For social welfare, these results indicate that when the platforms' discount factor increases from a low level to an intermediate level, social welfare will increase because the market moves from the equilibrium in which the low-quality platform wins, the one in which the high-quality platform overcomes its non-focal position and then maintain the focal position infinitely. However, the effect of a further increase in the discount factor on welfare is ambiguous because for a high discount factor there are multiple equilibria.

In our base model, the same platform wins in all periods. In some cases, platforms "take turns" in being the dominant platform. In the market for smartphones, for example, Nokia dominated the early stage, along with RIM, with smartphones based on physical keyboard. Apple then revolutionized the industry by betting on the new touch screen technology and its new operating system. Nokia and RIM stuck to their physical keyboard technology and operating systems in the subsequent updates of their products, and eventually lost the leadership position to Apple. Few years later, Samsung, managed to gain a substantial market share (though not strict dominance) by betting on smartphones with large screens, while Apple continues to stick to its 3.5 inch screen. Only recently, when it became evident that there is a high demand for smartphones with large screens, Apple decided to increase the screen size for iPhone 5. Today, Nokia is trying to regain its dominant position by betting on the new Windows Phone technology. In a bid to win back their position in the smartphone market, RIM (now simply called BlackBerry) introduced new phones, Q10 and Z10, with new operating system and many innovative features. Both Apple and Samsung choose to remain with their previous operating systems, and only offer periodical upgrades.

Such industry leader changes were also common in the history of the video-game consoles market. Nintendo, Sony and Microsoft alternated in being the market leader—none of them winning more than two generations in a row. While the technology significantly improved with each generation of video-game consoles, some generations were marked by radical innovation, e.g. Nintendo's Wii.[1]

To study such markets, we then consider the more realistic setting in which the platforms' qualities are stochastic: they vary in each period. Platforms observe the qualities at the beginning of each period, but are uncertain about the potential qualities in future periods. One of the platforms has a higher expected quality then the other. In equilibrium, each platform can win the market in each period with some probability. In particular, there is a threshold in the quality gap between the two platforms such that in each period, each platform wins the market if its quality is sufficiently higher in comparison with the quality of other platform.

We find that unlike the case of constant qualities, the expected social welfare under stochastic qualities can decrease the more platforms are long-sighted. In particular, social welfare when both platforms are substantially long-sighted (i.e., the discount factor is close to 1) is lower than in the case when both platforms are substantially short-sighted (i.e., the discount factor is close to 0). Supportingly, a platform can lose even if it is focal and offers a higher quality than the non-focal platform. This result can never emerge in the case of constant qualities.

The intuition for these results is that with stochastic qualities, a higher discount factor implies that a platform that expects to have high-quality realizations in future periods will have more of an incentive to compete aggressively to win the market in the current period, even if in the current period its quality is substantially inferior to that of the competing platform. This platform will win the market today more than it should from the viewpoint of maximizing expected social welfare, and is likely to maintain a focal position in future periods more than it should.

In a generalization to the concept of focal position, we consider intrinsic equilibrium uncertainty captured by assuming a public correlated equilibrium. Although being focal provides an advantage, it is limited in the sense that for a small value adjusted price differential the non-focal platform wins with positive probability. We find that while there may still be equilibria where the focal platform maintains forever its dominant position, existence conditions are more restrictive. Moreover there will be equilibria where alternation occurs

[1] Hagiu and Halaburda (2009)

in any period with positive probability.

Most theoretical analyses of platform competition focus on static games. Caillaud and Jullien (2001, 2003) consider competition between undifferentiated platforms, where one of them benefits from favorable beliefs. Hagiu (2006) considers undifferentiated platform competition in a setting where sellers join the platform first, and only then buyers. Lopez and Rey (2009) consider competition between two telecommunication networks when one of them benefits from customers' "inertia," such that in the case of multiple responses to the networks' prices, consumers choose a response which favors one of the networks. Halaburda and Yehezkel (2012) consider competition between platforms when one of them has only partial beliefs advantage. While all those papers acknowledge the dynamic nature of the platform competition, they aim at approximating the characteristics of the market in static models. Halaburda and Yehezkel (2012) explore how platform's strategies affect their future profits in a simple multi-period setup where the beliefs advantage depends on the history of the market. Markovich (2008) analyzes hardware standardization in a dynamic market where software firms invest in new product innovation. But the dynamics of platform competition is still underexplored. Cabral (2012) develops a dynamic model of competition with forward looking consumers but where only one consumer chooses at a time avoiding the coordination issue we focus on. Bligaiser, Crémer and Dobos (2013) consider dynamic competition in the presence of switching costs. Our model share with theirs the feature that success provides an incumbency advantage. But the intertemporal linkage and the demand dynamics differ between the two models. Moreover they consider switching cost heterogeneity, while we focus on quality differential. Argenziano and Gilboa (2012) consider a repeated coordination game where players use history to form beliefs regarding the behavior of other players. Our paper adopts the same approach in the context of platform competition and study how platforms should compete given such belief formation by consumers. In our paper, platforms can alter beliefs by wining and shifting consumers' coordination in their favor. Our paper is related to ongoing work by Biglaiser and Crémer (2012) trying to define a notion of consumer inertia creating an history dependency. We do not try to model how history dependency emerges but its implications for competition.

The rest of the paper is organized as follows. Section 2 describes the model. In Section 3, we consider the case where the platforms' qualities are constant in all periods. Section 4 considers the case of stochastic qualities. Section **??** generalizes the concept of focal platform to the case where consumers coordinate on joining each platform with some probability.

2 The Model

Consider an homogeneous population of size 1 and two competing platforms with the same cost normalized to 0. Time goes from $t = 0$ to infinity (but we will first consider finite horizon games). In the benchmark model the value of each platform is fixed over time, with value q_i at platform i. Consumers also benefit from network effects, the value other consumers joining same platform is β.[2] We assume that network effects are sufficiently strong that it is not possible to predict for sure which platform will emerge:[3]

Assumption: $\beta > |q_A - q_B|$

At any date the platform i sets a price p_{it}, and consumers decide which platform to join for the current period. In what follows prices can be negative, interpreted as price below cost.[4]

The issue with competition in an environment with network effects is that there is a multiplicity of equilibria. Indeed consider the allocation of consumers that emerges for given prices. If $q_i - p_{it} > q_j - p_{jt} + \beta$, then all consumers would join platform i. But if

$$|q_A - q_B + p_{Bt} - p_{At}| < \beta, \tag{1}$$

there are two possible allocations, all consumers join A or all join B. This multiplicity creates a major difficulty to discussing dynamic competition in environments with network effects, and several solutions have been proposed to address this issue. In this paper we rely on the idea of pessimistic beliefs and focal platform as developed in Caillaud-Jullien (2003), Hagiu (2006) and Jullien (2000). We say that platform i is *focal* if under condition (1), the consumers join platform i. We assume that at any date there is a focal platform.

Assumption: At any date there is a focal platform.

[2] Since the consumers are homogeneous they all join the same platform.

[3] The analysis of other case is direct and uninteresting.

[4] To allow for truly negative prices, we need to assume that agents who collect the subsidy indeed join the platform and provide the benefit to other users.

In this paper we focus on the dynamic game with T periods, $t = 0, \ldots, T-1$, allowing $T \to \infty$. A dynamic model allows to determine the identity of the focal platform in period $t > 0$ from the history. To simplify the matters, we focus on one period dynamics.

At every period t, let us summarize the market outcome by a pair (w_t, f_t), where $w_t \in \{A, B\}$ is the identity of the active platform, i.e., the platform who won the market in t,[5] and $f_t \in \{A, B\}$ is the identity of the focal platform in t. It is possible for the non-focal platform to win the market, therefore those two do not need to be the same. Based on the observation of past outcome, consumers form conjectures about the current period most likely to win. These conjectures are assumed to converge to a single focal platform. In $t = 0$ one of the platforms is arbitrarily set as the focal platform. Call this platform A. The dynamics of the platform focality is then given by transition probabilities

$$\Pr\left(f_t = i \mid w_{t-1}, f_{t-1}\right) .$$

At any date the focal platform f_t is common knowledge and it is the only payoff relevant variable. In what follows we concentrate on *Markov Perfect equilibria* where the state variable is f_t. A (Markov) equilibrium is characterized by equilibrium prices p_i^f, for all i and f, interpreted as the equilibrium price of platform i when the focal platform is f.

In the base model we consider a deterministic rule where the last winner of the market becomes focal, i.e.,

$$\Pr\left(f_t = w_{t-1} \mid w_{t-1}, f_{t-1}\right) = 1 .$$

In the extensions, we consider other rules, also non-deterministic.

3 Constant Qualities

In this section we consider the case where the qualities of the two platforms are constant in all periods. The main conclusion of this section is that a low-quality platform can take advantage of its focal position for wining the market only if platforms are sufficiently short-sighted (low δ). Otherwise (intermediate and high δ), the high-quality platform can overcome a non-focal position by winning the market and then maintaining the focal position infinitely. As a result, an increase in δ can be welfare enhancing.

[5] In this model there cannot be market sharing in equilibrium: at each date, a single platform attracts the whole population.

It is straightforward to see that in a one period game with focal platform i, the only equilibrium is

$$p_i = q_i - q_j + \beta,\ p_j = 0$$

and consumers join the focal platform.

3.1 Finite Horizon

Consider a two period case, i.e., $T = 2$. Since the subgame in the last period is a one-period game, the focal platform, say i, earns $q_i - q_j + \beta$, and the non-focal earns 0. Thus, in $t = 0$, the platform that wins and becomes focal in the last period earns additional $\delta(q_i - q_j + \beta)$. To win in $t = 0$, the focal platform A needs to set $p_{A0} \leq p_{B0} + q_A - q_B + \beta$. Platform B needs to set $p_{B0} < p_{A0} + q_B - q_A - \beta$. When platform i makes a sale in period t, it receives revenue $\pi_{it} = p_{it}$. Wen it does not make a sale, $\pi_{it} = 0$, independently of p_{it}. If platform B wins $t = 0$, its total payoff is $\pi_{B0} + \delta\pi_{B1} = p_{B0} + \delta(q_B - q_A + \beta)$. Therefore, platform B never sets p_{B0} lower than $-\delta(q_B - q_A + \beta)$. To win the market platform A sets p_{A0} such that it would force $p_{B0} < -\delta(q_B - q_A + \beta)$, i.e.,

$$p_{A0} = -\delta(q_B - q_A + \beta) + q_A - q_B + \beta\,, \tag{2}$$

and earns

$$\Pi_A(T = 2) = p_{A0} + \delta(q_A - q_B + \beta) = (1 + 2\delta)(q_A - q_B) + \beta\,. \tag{3}$$

It is only worth for platform A to win the market when $\Pi_A(\textit{two periods}) \geq 0$. When $(1 + 2\delta)(q_A - q_B) + \beta < 0$, platform B sets $p_{B0} = -\delta(q_A - q_B + \beta) + q_B - q_A - \beta$, wins the market in $t = 0$ and earns profits

$$\Pi_B(T = 2) = p_{B0} + \delta(q_B - q_A + \beta) = (1 + 2\delta)(q_B - q_A) - \beta > 0\,,$$

which are positive.

Consider now a game with a longer horizon, i.e., $T > 2$. Suppose first that $q_A > q_B$. It is intuitive that platform A wins the market in every period as it is both focal and more efficient. We show that this is also the case even for some $q_A < q_B$ for short time horizons.

More precisely, let us denote by $\Pi_i^f(T)$ the discounted profit of platform i when platform f is focal in date $t = 0$ and there are T periods. Then, in a T period game, platform i is willing to sacrifice $\delta\Pi_i^i(T-1)$ in the current period to win the market. Thus, to win in $t=0$, the focal platform A needs to set

$$p_{A0} \le p_{B0} + q_A - q_B + \beta = -\delta\Pi_B^B(T-1) + q_A - q_B + \beta\,.$$

Then, platform A's profit is[6]

$$\Pi_A(A \text{ wins in } t=0) = q_A - q_B + \beta - \delta\Pi_B^B(T-1) + \delta\Pi_A^A(T-1) = (q_A - q_B)\sum_{k=0}^{T-1}(2\delta)^k + \beta\,.$$

Similarly, platform B's profit if it wins in $t=0$ is

$$\Pi_B(B \text{ wins in } t=0) = (q_B - q_A)\sum_{k=0}^{T-1}(2\delta)^k - \beta \equiv -\Pi_A(A \text{ wins in } t=0)\,.$$

Notice that only for one of the platforms the payoff from winning the market, $\Pi_i(i \text{ wins in } t = 0)$, is positive. Therefore, for only one of the platforms it is worthwhile to win and hold the market.

At this stage we may distinguish two cases. The first case is when $\Pi_i^i(T)$ derived above is positive for $i = A, B$ and all T. This occurs if $q_A = q_B$, or if $\delta < 1/2$ and

$$|q_A - q_B| < \beta(1-2\delta)\,.$$

In this case the initial focal platform A wins and keep the market at all dates with profit Π_A^A.

In the other case we have

$$|q_A - q_B| \ge \beta(1-2\delta)\,.$$

Suppose that $q_A - q_B < 0$. Let T_A be the largest T such that $\Pi_A^A(T)$ determined above is positive. Then platform A wins the market whenever $T \le T_A$ while $\Pi_A^A(T) = 0$ and platform B wins the market if $T > T_A$. Suppose now that $q_A - q_B > 0$. Then we can define T_B as the largest T such that $\Pi_B^B(T)$ determined above is positive. Then platform A wins the market in any equilibrium but the profit depends on T. As long as $T \le T_B + 1$, the above reasoning

[6] The last equality follows from applying the same formulas recursively in

$$\Pi_A^A(T-1) - \Pi_B^B(T-1) = 2(q_A - q_B) + 2\delta[\Pi_A^A(T-2) - \Pi_B^B(T-2)] = 2(q_A - q_B)\sum_{k=0}^{T-2}(2\delta)^k\,.$$

applies and the profit is given by 3.1. But when $T > T_B + 1$,the platform B is not willing to make any sacrifice until the horizon is $T_B + 1$. Hence the profit of platform A for $T > T_B + 1$ is

$$\Pi_A^A(T) = (q_i - q_j) \sum_{k=0}^{T-1} (2\delta)^k + \beta \sum_{k=0}^{T-T_B-1} (\delta)^k .$$

Remark. For $\delta \neq \frac{1}{2}$, $\sum_{k=0}^{T-1} (2\delta)^k = \frac{1-(2\delta)^T}{1-2\delta}$. For $\delta = \frac{1}{2}$, the sum equals T. Notice that the sum is never negative.

3.2 Infinite Horizon: Extrapolating $T \to \infty$

If we extrapolate platform A's profit to $T \to \infty$, we conclude that when platform A wins the market, it earns the profit

$$\Pi_A(T \to \infty) = (q_A - q_B) \sum_{k=0}^{\infty} (2\delta)^k + \beta .$$

For $\delta < \frac{1}{2}$, $\sum_{k=0}^{\infty} (2\delta)^k = \frac{1}{1-2\delta}$, which yields $\Pi_A(T \to \infty | \delta < \frac{1}{2}) = \frac{q_A - q_B}{1-2\delta} + \beta$. When

$$q_A - q_B \geq -\beta(1 - 2\delta) ,$$

platform A wins the market and wins positive profits. Otherwise platform B wins and earns positive profits.

This, of course, poses questions about what happens in the market—what are the strategies, who wins—when the future is valued highly, $\delta \geq \frac{1}{2}$. For patient platforms, our above results show that platform A wins the market only when $q_A - q_B \geq 0$. When $q_A - q_B < 0$ and $\delta \geq \frac{1}{2}$, platform A would never find it worthwhile to win the market, if winning the market would require setting $p_A^t < 0$ at any t. Then, the best response of platform B is to set $p_B^{t=0} = q_B - q_A - \beta$ in period 0 to win the market and become focal, and $p_B^t = q_B - q_A + \beta$ in all subsequent periods.

Therefore, when $\delta \geq \frac{1}{2}$, the better quality platform wins the market (independently on which one is focal in $t = 0$), and earns infinite profits.

3.3 Infinite Horizon: Other Equilibria

Accounting for an infinite horizon game may give rise to other equilibria than extrapolating the finite game to $T \to \infty$.[7] Every period t of the infinite game is characterized by the state variable at time t, f_t. The equilibrium is characterized by the strategies of both platforms in all possible states, and the outcome in each state. We will consider three pure strategy equilibria outcomes: (i) platform A wins in both states, (ii) platform B wins in both states, and (iii) the focal platform wins.[8]

In what follows we characterize the strategies supporting those equilibria outcomes, and conditions under each equilibrium exists. We define the value function V_i^f as the expected discounted profit of platform i when platform f is focal.

Consider first the equilibrium outcome where platform A wins in both states. In this equilibrium the value functions for platform B are $V_B^B = V_B^A = 0$. In such an equilibrium, platform B always sets price $p_B = 0$. In no situation platform B would like to set $p_B^f < 0$, because it cannot count on future profits to justify the "investment" in taking over the market. When A is focal, it optimally sets $p_A^A = q_A - qB + \beta$. Similarly, were B focal, platform A sets $p_A^B = qA - qB - \beta$, and platform B sets $p_B^B = 0$. Were A to set a higher price, platform B would keep the market and make non-negative profits. In such a case

$$V_A^A = q_A - q_B + \beta + \delta V_A^A \qquad \text{and} \qquad V_A^B = q_A - q_B - \beta + \delta V_A^A \,.$$

Moreover, incentive compatibility for platform A requires that

$$V_A^A \geq \delta V_A^B \qquad \text{and} \qquad V_A^B \geq 0\,.$$

Therefore, this equilibrium exists when $q_A - q_B \geq \beta(1 - 2\delta)$. With a similar analysis for platform B, we arrive at the following result.

Lemma 1 *There is an equilibrium where platform i wins in both states if $q_i - q_j \geq \beta(1-2\delta)$.*

Lemma 1 shows that a non-focal platform B can win a focal position and maintain it in all future periods in the following cases. First, when its quality is substantially superior

[7] One reason for it may be that in a finite game we use subgame perfect equilibrium concept, and in the truly infinite game, we use Markov perfect equilibrium concept.

[8] The fourth possibility of a pure strategy equilibrium outcome: that non-focal platform wins cannot be supported by any strategy.

than the quality of platform A. Second, when platforms are very forward-looking, such that δ is high. Third, when β is low.

The remaining equilibrium to consider is one where the focal platform wins. Recall that p_i^f denotes the price of platform i when f is focal in such an equilibrium. Since the wining platform anticipates it will stay active and focal from the new period on, we have values function

$$V_i^i = \frac{p_i^f}{1-\delta}, \quad V_i^j = 0$$

The benefits of selling at a given date is $p_{it} + \delta V_i^i$. It follows that the minimal profit that platform i is willing to sacrifice today to gain the market is $-\delta V_i^i$. In such an equilibrium it must be the case that the focal platform sets a price $p_i^i \leq q_i - q_j + \beta - \delta V_j^j$, otherwise the competing platform would set a price above $-\delta V_i^i$ and wins the market. Ruling out cases where $p_j < -\delta V_j^j$ because this is weakly dominated for firm j, we obtain equilibrium prices[9]

$$p_i^i = q_i - q_j + \beta - \delta V_j^j, \; p_j^i = -\delta V_j^j.$$

This leads to values function in such an equilibrium solutions of

$$\begin{aligned} (1-\delta) V_A^A + \delta V_B^B &= q_A - q_B + \beta \\ (1-\delta) V_B^B + \delta V_A^A &= q_B - q_A + \beta \end{aligned}$$

yielding

$$V_A^A = \frac{q_A - q_B}{1-2\delta} + \beta; \quad V_B^B = \frac{q_B - q_A}{1-2\delta} + \beta.$$

We then conclude that:

Lemma 2 *There is an equilibrium where the focal platform wins in every state if* $\beta\left|1-2\delta\right| > \left|q_B - q_A\right|$.

Proof. For this to be an equilibrium it is necessary and sufficient that $V_A^A > 0$ and $V_B^B > 0$. ■

Lemma 2 shows that a focal platform A can maintain its focal position in all future periods even when it offers a lower quality than platform B. To see the intuition for this

[9]This is innocuous for existence argument

result, consider first the case of $\delta \leq 1/2$. As Lemma 2 shows, the equilibrium holds in this case if δ and the quality gap, $q_B - q_A$, are sufficiently low. Intuitively, suppose that q_B increases. This has two effects on the equilibrium V_B^B. First, a *direct* effect – since $p_B^B = q_B - q_A + \beta - \delta p_A^B$, taking V_A^A as given, platform B can now attract agents with a higher p_B^B, implying that V_B^B will increase. Second, a *strategic* effect – since $p_B^A = -\delta V_B^B$, platform A will now know that even if it is focal, it will compete against a more aggressive platform B, because platform B has more to gain by becoming focal. This reduces V_A^A, which in turn increases V_B^B because platform A will not compete aggressively to gain a focal position when it is not focal. Both the direct and the strategic effects work in the same direction of increasing V_B^B and decreasing V_A^A. If the gap $q_B - q_A$ is sufficiently wide, $V_A^A < 0$, implying that platform A cannot maintain its focal position when competing against a superior quality platform. As δ increases, platform B cares more about future profit so it will have a stronger incentive to win the market when it is not focal, and maintain its focal position when it if focal.

Now suppose that $\delta > 1/2$. As Lemma 2 reveals, in this case the equilibrium is completely reversed. Now, if $q_B > q_A$, then $V_A^A > V_B^B$, and as q_B increases, V_B^B decreases while V_A^A increases. However, the equilibrium in the case of $\delta > 1/2$ relays on the somewhat strong assumption that platforms "overreact", such that as q_B increases, while the direct effect increases V_B^B (as in the case of $\delta < 1/2$), the strategic effect works in the opposite direction and is stronger than the direct effect. To see how, suppose that platform B is focal and q_B increases. The equilibrium holds when platform B expects that as a response to the increase in q_B, platform A will over-react in the opposite direction than in the case of $\delta < 1/2$, by becoming very aggressive and decreasing its price when it is not focal, p_A^B. In this case, V_B^B increases since q_B increases (direct effect), but decreases since p_A^B decreases (strategic effect), but since the future is very important, the strategic effect outweighs the direct effect, and overall effect is to decrease V_B^B and as a result to increase V_A^A.

Notice that if we rule out the possibility of overreaction, then the equilibrium in which the focal platform A always win fails when $q_B > q_A$ and $\delta > 1/2$. The equilibrium in which the non-focal platform B wins in the first period and maintain its focal position infinitely, as we characterized is Lemma 1, always holds when $q_B > q_A$ and $\delta > 1/2$, and does not relay on platforms' overreactions.

Proposition 1 below summarizes the results of Lemma 1 and Lemma 2:

Proposition 1 (equilibria under constant qualities) *Suppose that platform A is focal at period t=0. Then,*

(i) for $q_B - q_A > \beta|1-2\delta|$ there exists a unique equilibrium, and in that equilibrium platform B wins;

(ii) for $\beta(1-2\delta) < q_B - q_A < \beta|1-2\delta|$, which occurs only for $\delta > 1/2$, there exist multiple equilibria, and in one of those equilibria platform B wins;

(iii) for $q_B - q_A < \beta(1-2\delta)$, platform A wins in all equilibria.

Proof. This follows from the assumption that A is initially focal and from Lemma 1 and Lemma 2. ■

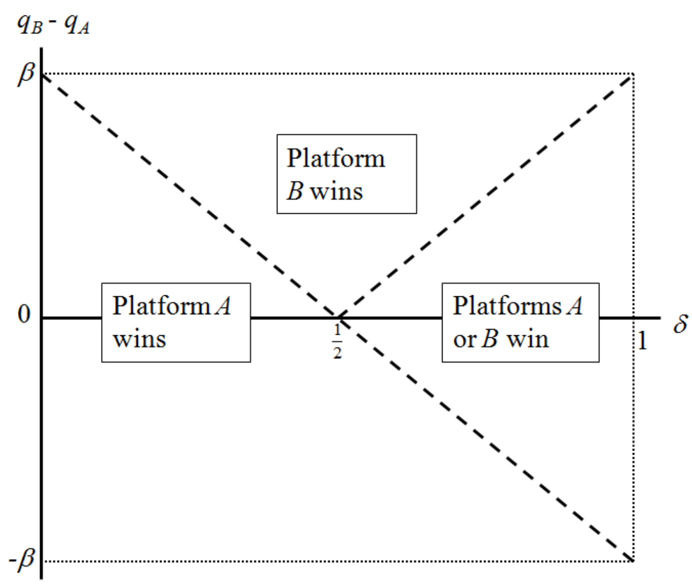

Figure 1: Equilibrium configuration

The equilibrium active platform is depicted in Figure 1. The figure shows that for low values of δ and $q_B - q_A$, there is a unique equilibrium in which a low-quality focal platform A wins. Intuitively, in this case the same qualitative results of a static game follows to the dynamic game. For intermediate values of δ and high values of $q_B - q_A$, there is a unique equilibrium in which the non-focal platform B takes over the market and maintain its focal position infinitely. For high values of δ and low values of $q_B - q_A$, there are multiple equilibria in which either platform A or B win.

4 Stochastic qualities

The previous section focused on the case where the qualities of the two platforms are constant for infinity. Consequently, in any equilibrium the same platform wins the market in all periods. In many markets for platforms there is a shift in leadership every few years, parallel to technology improvements. In this section we consider the more realistic case in which qualities are stochastic. We show that there is an equilibrium in which each platform can win in each period with some probability. The main conclusion of this section is that unlike the constant-qualities case, social welfare under stochastic qualities may decrease with δ.

Suppose that qualities change randomly in each period. At the beginning of each period, both platforms observe the realization of their qualities for this particular period. Then, the two platforms compete by setting prices.

The results of Section 3 showed that the equilibrium depends on the difference between the qualities of the two platforms, and not their absolute values. Suppose then, without loss of generality, that $q \equiv q_B - q_A$ change randomly in each period with full support on the real line according to a probability function $f(q)$, with a cumulative distribution function $F(q)$. Our assumption that the support is infinite ensures that there is an equilibrium in which each platform can win the market with a positive probability.[10] Suppose that q has a mean of μ and a variance σ^2. We focus on the case where $\mu > 0$ such that on average, platform B is of superior quality than platform A. The case of $\mu < 0$ is symmetric.

Suppose that platform A is focal in period $t = 0$. Consider an equilibrium in which there are two cutoffs, $\bar{q}^A$ and $\bar{q}^B$, such that if platform A is focal in time t, it wins if $q \leq \bar{q}^A$ and platform B wins otherwise. Likewise, if platform B is focal in time t, it wins if $q \geq \bar{q}^B$ and platform A wins otherwise. This equilibrium has the feature that given that platform A starts at $t = 0$ as the focal platform, it will win in every period as long as $q < \bar{q}^A$. Then, once there is a realization with $q > \bar{q}^A$, platform B takes over the market and becomes focal. Platform B will maintain its focal position in future periods as along as $q \geq \bar{q}^B$, until eventually in a certain period there is a realization of q with $q < \bar{q}^B$, and platform A wins back its focal position. The game then repeats itself infinitely, with platforms "taking turns" in winning depending on the realization of q.

[10]We should note that this is a stronger assumption than what we need, as our results hold even with a finite support, as long as it is wide enough. Our assumption of infinite support facilitates the analysis and enables us to avoid corner solutions.

Let V_i^f denote the expected value function of platform i when platform f is focal. To solve for the equilibrium, suppose that platform A is focal in time t and the quality difference has some realization, q. The lowest price platform B is willing to charge in order to win the market is $-\delta V_B^B + \delta V_B^A$. This is because platform B will earn the expected value of V_B^B from becoming focal in the next period, and earn the expected value of V_B^A from remaining non-focal. To win the market faced to this price, the focal platform A will need to set $p_A = \beta - q - \delta V_B^B + \delta V_B^A$. Platform A earns $p_A + \delta V_A^A$ if indeed it wins (if $q \leq \bar{q}^A$) and earns $0 + \delta V_A^B$ if it loses (if $q > \bar{q}^A$). Therefore:

$$V_A^A = \int_{-\infty}^{\bar{q}^A} (\beta - q - \delta V_B^B + \delta V_B^A + \delta V_A^A) f(q) dq + \int_{\bar{q}^A}^{\infty} \delta V_A^B f(q) dq.$$

Suppose now that platform A is non-focal. The lowest price platform B is willing to charge to maintain its focal position is $p_B^B = -\delta V_B^B + \delta V_B^A$. Again, if platform A wins, it sets p_A^B that ensures that $-p_A^B \geq \beta - p_B^B + q$, or $p_A^B = -\beta - q - \delta V_B^B + \delta V_B^A$. Platform A earns $p_A^B + \delta V_A^A$ if indeed it wins the market (when $q \leq \bar{q}^B$), and earns $0 + \delta V_A^B$ if it loses the market (when $q > \bar{q}^B$). Therefore:

$$V_A^B = \int_{-\infty}^{\bar{q}^B} (-\beta - q - \delta V_B^B + \delta V_B^A + \delta V_A^A) f(q) dq + \int_{\bar{q}^B}^{\infty} \delta V_A^B f(q) dq.$$

The cases of V_B^B and V_B^A are symmetric by recalling that platform B wins the market if $q \geq \bar{q}^B$ when it is focal, and if $q > \bar{q}^A$ when it is not. Moreover, q positively affects the profit of platform B. Therefore:

$$V_B^B = \int_{\bar{q}^B}^{\infty} (\beta + q - \delta V_A^A + \delta V_A^B + \delta V_B^B) f(q) dq + \int_{-\infty}^{\bar{q}^B} \delta V_B^A f(q) dq,$$

$$V_B^A = \int_{\bar{q}^A}^{\infty} (-\beta + q - \delta V_A^A + \delta V_A^B + \delta V_B^B) f(q) dq + \int_{-\infty}^{\bar{q}^A} \delta V_B^A f(q) dq.$$

Next consider the equilibrium $\bar{q}^A$ and $\bar{q}^B$. The equilibrium $\bar{q}^A$ is such that for $q = \bar{q}^A$, a focal platform A is exactly indifferent between wining the market or not, taking the equilibrium future value functions and the price of platform B as given. That is:

$$\beta - \bar{q}^A - \delta V_B^B + \delta V_B^A + \delta V_A^A = \delta V_A^B.$$

Notice that the condition for making the non-focal platform B indifferent between winning and not is equivalent to the condition above. Turning to $\bar{q}^B$, the equilibrium $\bar{q}^B$ should be such that for $q = \bar{q}^B$, a non-focal platform A is exactly indifferent between wining the market or not, taking the equilibrium future value functions and the price of platform B as given. That is:

$$-\beta - \bar{q}^B - \delta V_B^B + \delta V_B^A + \delta V_A^A = \delta V_A^B .$$

Again notice that the condition for making the focal platform B indifferent between winning and not is equivalent to the condition above.

The set of the six equations above define the equilibrium V_A^A, V_A^B, V_B^B, V_B^A, $\bar{q}^A$ and $\bar{q}^B$. Using the above equations, the following proposition describes the features of the equilibrium $\bar{q}^A$ and $\bar{q}^B$:

Proposition 2 (The effect of δ on $\bar{q}^A$ and $\bar{q}^B$) *Suppose that $4\beta \max f(q) < 1$. There are unique equilibrium values of $\bar{q}^A$ and $\bar{q}^B$, with the following features:*

(i) for $\delta = 0$, $\bar{q}^A = \beta$ and $\bar{q}^B = -\beta$;

(ii) $\bar{q}^A$ and $\bar{q}^B$ are decreasing with δ;

(iii) $\bar{q}^A - \bar{q}^B = 2\beta$ for all δ;

(iv) if δ is high enough and $F(0) < 1/4$, then $\bar{q}^A < 0$;

Proof. Directly from the formulas for V_A^A, V_A^B, V_B^B, V_B^A, and conditions for $\bar{q}^A$ and $\bar{q}^B$, we obtain

$$\bar{q}^A - \bar{q}^B = 2\beta .$$

Moreover,

$$\begin{aligned}
V_A^A &= \int_{-\infty}^{\bar{q}^A} \left(\bar{q}^A - q\right) f(q)\, dq + \delta V_A^B \\
V_A^B &= \int_{-\infty}^{\bar{q}^B} \left(\bar{q}^B - q\right) f(q)\, dq + \delta V_A^B = \frac{1}{1-\delta} \int_{-\infty}^{\bar{q}^B} \left(\bar{q}^B - q\right) f(q)\, dq
\end{aligned}$$

and

$$\begin{aligned}
V_B^B &= \int_{\bar{q}^B}^{+\infty} \left(q - \bar{q}^B\right) f(q)\, dq + \delta V_B^A \\
V_B^A &= \frac{1}{1-\delta} \int_{\bar{q}^A}^{+\infty} \left(q - \bar{q}^A\right) f(q)\, dq
\end{aligned}$$

The optimality condition is then

$$\begin{aligned}\bar{q}^A &= \beta - \delta V_B^B + \delta V_B^A + \delta V_A^A - \delta V_A^B \\ &= \beta - \delta \int_{\bar{q}^B}^{+\infty} \left(q - \bar{q}^B\right) f(q)\, dq + \delta \int_{\bar{q}^A}^{+\infty} \left(q - \bar{q}^A\right) f(q)\, dq + \delta \int_{-\infty}^{\bar{q}^A} \left(\bar{q}^A - q\right) f(q)\, dq - \delta \int_{-\infty}^{\bar{q}^B} \left(\bar{q}^B - q\right) f(q)\, dq \\ &= \beta + \delta \phi\left(\bar{q}^A\right)\end{aligned}$$

where

$$\phi\left(\bar{q}^A\right) = \int_{\bar{q}^A}^{+\infty} \left(q - \bar{q}^A\right) f(q)\, dq + \int_{-\infty}^{\bar{q}^A} \left(\bar{q}^A - q\right) f(q)\, dq - \int_{-\infty}^{\bar{q}^B} \left(\bar{q}^B - q\right) f(q)\, dq - \int_{\bar{q}^B}^{+\infty} \left(q - \bar{q}^B\right) f(q)\, dq$$

or

$$\phi\left(\bar{q}^A\right) = \int_{\bar{q}^A}^{+\infty} (-2\beta) f(q)\, dq + \int_{\bar{q}^A - 2\beta}^{\bar{q}^A} \left(2\bar{q}^A - 2\beta - 2q\right) f(q)\, dq + \int_{-\infty}^{\bar{q}^A - 2\beta} (2\beta) f(q)\, dq$$

We have

$$\begin{aligned}\phi'\left(\bar{q}^A\right) &= 2\left(F\left(\bar{q}^A\right) - F\left(\bar{q}^A - 2\beta\right)\right) \\ \phi(-\infty) &= -2\beta \qquad (4) \\ \phi(+\infty) &= 2\beta \qquad (5)\end{aligned}$$

We notice that $\phi'\left(\bar{q}^A\right) < 1$ when

$$2\delta \max\left(F(q) - F(q - 2\beta)\right) < 1.$$

In this case the equilibrium is unique. This is the case for all δ and if $4\beta \max f(q) < 1$.

Then $\bar{q}^A$ is less than 0 if

$$0 > \beta + \delta\phi(0)$$

which holds for δ large if

$$-\beta > \phi(0) = -2\beta\left(1 - 2F(-2\beta)\right) + \int_{-2\beta}^{0} (-2q) f(q)\, dq = -2\beta + 2\int_{-2\beta}^{0} F(q)\, dq$$

or if

$$\beta > 2\int_{-2\beta}^{0} F(q)\, dq$$

This is true for all β if $F(0) < 1/4$. ■

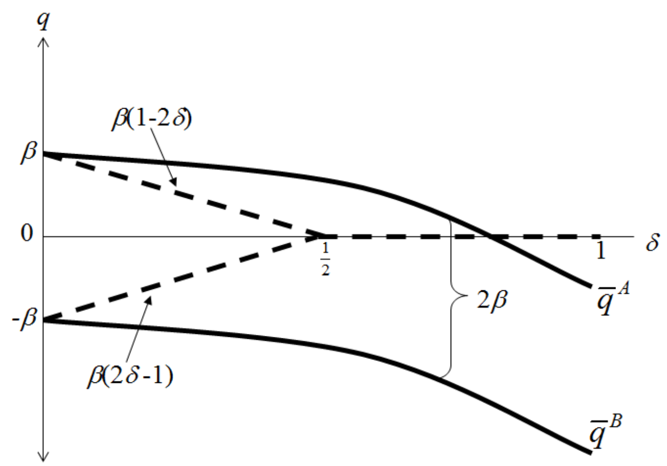

Figure 2: The effect of δ on $\bar{q}^A$ and $\bar{q}^B$

The results of Proposition 2 are illustrated in Figure 2. Recall that when qualities are constant in all periods and platforms do not overreact, a focal platform A wins if $q < \max\{\beta(1-2\delta), 0\}$ while a focal platform B wins if $q > \min\{-\beta(1-2\delta), 0\}$ (the two dashed lines in Figure 2). As a result, an increase in δ makes it easier for the highest quality platform to win the market. With stochastic qualities, however, Proposition 2 reveals that an increase in δ does not necessarily increases the probability that the highest quality platform wins. Where platform B is focal, we find that $\bar{q}^B < 0$ and decreases with δ. Therefore, a focal platform B is more likely to win the market with a lower quality than platform A as δ increases. When platform A is focal and δ is sufficiently close to 0, platform A wins even when it offers a lower quality than platform B. As δ increases, it is less likely that the focal platform A will be able to maintain it's focal position with a lower quality than platform B, in that $\bar{q}^A$ decreases with δ. However, if δ is sufficiently high, then $\bar{q}^A$ crosses the 0 line and becomes negative. In this case, platform A can lose the market even if it is focal and of superior quality than platform B (for realizations $\bar{q}^A < q < 0$). The cutoff $\bar{q}^A$ decreases further below 0 as δ increases.

The intuition for these results is the following. Recall that a platform's expected profit includes current profit and the probability of maintaining its focal position in future periods. Since $\mu > 0$, platform B is more likely to have in future periods a higher quality than platform A. As δ increases, platform A internalizes that it is less likely to win in future periods and will therefore have less of an incentive to compete aggressively in the current period. Platform B internalizes that it is more likely to win the market in future periods and will therefore have more of an incentive to compete aggressively to win the market in the current period. This in turn provides platform B with a stronger competitive advantage over platform A, even when the current quality of platform B is inferior than the quality of platform A.

Next we turn to social welfare. We ask whether social welfare increases or decreases as platforms become more patient (i.e., δ increases). To this end, we normalize $q_A = 0$ and therefore $q_B = q$. Let W^i, $i = A, B$, denote the recursive expected social welfare when platform i is focal in period t, where:

$$W^A = \int_{-\infty}^{\bar{q}^A} (\beta + \delta W^A) f(q) dq + \int_{\bar{q}^A}^{\infty} (\beta + q + \delta W^B) f(q) dq,$$

$$W^B = \int_{\bar{q}^B}^{\infty} (\beta + q + \delta W^B) f(q) dq + \int_{-\infty}^{\bar{q}^B} (\beta + \delta W^A) f(q) dq.$$

We focus on the simplifying assumption that q is uniformly distributed along the interval $[\mu - \sigma, \mu + \sigma]$. The parameter σ measures the spread of quality realizations. Under uniform distribution, the two cutoffs $\bar{q}^A$ and $\bar{q}^B$ are:

$$\bar{q}^A = \beta - \frac{2\delta\mu\beta}{\sigma - 2\delta\beta}, \quad \bar{q}^B = -\beta - \frac{2\delta\mu\beta}{\sigma - 2\delta\beta}.$$

To ensure that $\bar{q}^A < \mu + \sigma$ and $\bar{q}^B > \mu - \sigma$, suppose that σ is high enough such that $\sigma > \frac{1}{2}(\mu + 3\beta) + \frac{1}{2}\sqrt{(\mu^2 + 6\mu\beta + \beta^2}$. Notice that this assumption implies that $\sigma > 2\beta$. With these assumptions, we can obtain the following comparison:

Proposition 3 (The effect of δ on social welfare) *Suppose that q is uniformly distributed along the interval $[\mu - \sigma, \mu + \sigma]$. Then,*

(i) evaluated at $\delta = 0$, $W^A = W^B$ and W^A is increasing with δ while W^B is decreasing with δ ;

(ii) $W^A > W^B$ for $\delta \in (0, 1)$;

(iii) evaluated at $\delta = 1$, $W^A = W^B$ and their values are lower than at $\delta = 0$.

Proposition 3 shows that when δ is sufficiently low (close to 0), social welfare increases with δ when platform A is focal and decreases with δ when platform B is focal. When δ is sufficiently high (close to 1), social welfare is lower than in the static case regardless of the identity of the focal platform. To obtain further insights, Figure 3 plots W^A and W^B as a function of δ for $\beta = 1$, $\sigma = 4$ and $\mu = 1$ (other numerical values obtain qualitatively similar results). As the figure reveals, when platform A is focal, social welfare is first increasing and then deceasing with δ. When platform B is focal, social welfare is strictly decreasing with δ. These results indicate that unlike the case of constant qualities, with stochastic qualities social welfare can decrease when platforms become more patient.

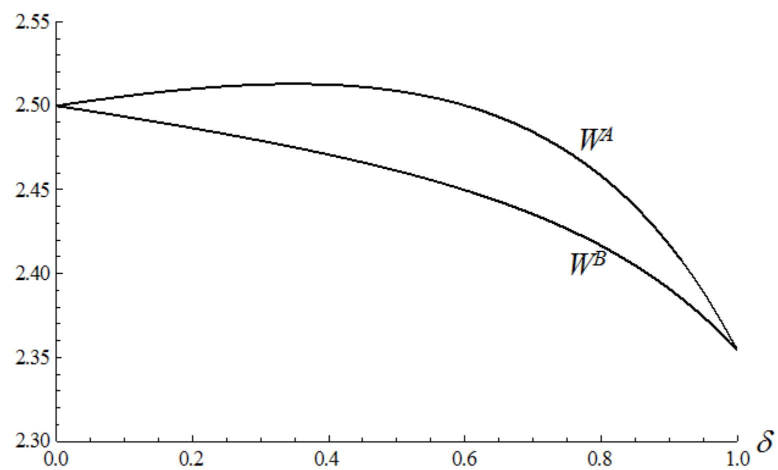

Figure 3: The effect of δ on social welfare

The intuition for these results are the following. Consider first the case where platform B is focal. As Proposition 2 shows, $\bar{q}^B < 0$ for $\delta = 0$, and $\bar{q}^B$ decreases with δ. Since optimum social welfare requires that platform B wins only when $q > 0$, social welfare when platform B is focal decreases because of two effects that work in the same direction. First, in the current period, a decrease in $\bar{q}^B$ below 0 implies that platform B has a higher probability of winning when it should not: $F(0) - F(\bar{q}^B)$ increases. Second, it implies that platform B has a higher probability of maintaining its focal position and continue winning (again, when it should not win) in the future. Next suppose that platform A is focal. Now, the two effects work in opposite directions. For $\delta = 0$, $\bar{q}^A > 0$ and $\bar{q}^A$ decreases with δ. As a result, in the current period it is less likely that platform A wins when it should not: $F(\bar{q}^A) - F(0)$ decreases. This effect increases social welfare. However, it is more likely that platform A

will lose its focal position in the future to platform B. This effect decreases social welfare. If δ is sufficiently small, the first effect dominates because the future is not very important so δ increases social welfare. If however δ is sufficiently high the second effect dominates and social welfare decreases with δ.

5 Conclusions

In platform competition, having a superior quality over a competing platform may not be enough to dominate the market. In the presence of strong network externalities, a platform needs to convince consumers that other consumers will join it. A platform can benefit from a focal position such that when consumers decide which platform to join, and multiple equilibrium decisions exist, all consumers play the equilibrium in which they join the focal platform. Network externalities provide the focal platform with a competitive advantage, while it is a disadvantage for a non-focal platform. In a static model a platform can use its focal position for dominating the market even when it offers lower quality than the non-focal platform. The aim of our paper is to study whether this static advantage carries over to a dynamic environment.

In this paper we consider the two sources of platforms' competitive advantage: quality and focal position, in a dynamic, infinite-horizon model. Two platforms differ in their qualities. In the first period, the inferior quality platform is focal, but platforms take into account that dominating the market in a current period provides the platform with a focal position in the next period.

We first consider an environment where qualities are constant in all periods. We find that when platforms are impatience, a focal but low-quality platform can maintain its focal position even with lower quality than a non-focal platform. However, when platforms are forward-looking, there can be multiple equilibria in which either the low or the high quality platform win.

We then turn to an environment with stochastic qualities which allows each platform win every period with some probability. Now, the more platforms are forward looking, it is less likely that a high-quality platform will be able to overcome a non-focal position. Intuitively, if one of the platforms has a higher average quality than the other platform, dynamic consideration will provide it with a stronger incentive to win the market and maintain a focal position in the current period, even with an inferior quality. At an extreme case, a focal and high-quality platform can still lose the market, if platforms are sufficiently forward looking. This result indicates that with stochastic qualities, social welfare can decrease the more platforms are forward looking.

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